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Ehresmann connections for lagrangian foliations

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Abstract

The notion of an Ehresmann connection was introduced by Ehresmann (1950). In recent years it has been extensively studied by some authors. The main aim of this paper is to demonstrate that under some relatively natural assumptions lagrangian foliations admit Ehresmann connections.

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1. Preliminaries

In this section, for the convenience of the reader, we will recall some basic definitions and results. The only new result of this section is Proposition 2.

1.1. Totally geodesic foliations

A foliation on a complete Riemannian manifold (M, g) is called totally geodesic if its leaves are totally geodesic submanifolds of (M, g).

Let Q be the orthogonal complement of $T\mathcal{F}$; it defines a natural splitting of the tangent bundle $TM = T\mathcal{F} \oplus Q$.

Let us fix a point $x \in M$. For any pair of curves $\alpha : [0, a] \to M$ and $\beta : [0, b] \to M$ with the same starting point, i.e. $\alpha(0) = \beta(0)$, such that the curve α is tangent to the leaf passing through the point x, i.e. it is a leaf curve, and β is tangent to Q, i.e. it is orthogonal to the foliation, there exists a smooth mapping $\sigma : [0, a] \times [0, b] \to M$ such that:

(1) the curves $\sigma_s : [0, a] \to M$, $\sigma_s = \sigma \mid [0, a] \times \{s\}$, $s \in [0, b]$, are contained in the corresponding leaves of $\mathcal{F}$;

(2) $\sigma_0 = \alpha$;

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(3) the curves $\sigma^t : [0,b] \to M$, $\sigma^t = \sigma \mid \{t\} \times [0,b]$, $t \in [0,a]$, are orthogonal to the foliation;

(4) $\sigma_0 = \beta$.

The existence of the smooth mapping σ allows us to push any orthogonal curve β along the curve α and any leaf curve α along the curve β, namely

$$\beta_\sharp : \alpha \mapsto \beta_\sharp\alpha = \sigma_b \quad \text{and} \quad \alpha_\sharp : \beta \mapsto \alpha_\sharp\beta = \sigma^a.$$

They map homotopic paths into homotopic ones, and these mappings depend on the homotopy class of the curve, i.e. if β is homotopic to the curve β' relative to its ends, then $\beta_\sharp = \beta'_\sharp$. Therefore any orthogonal curve $\beta : [0,b] \to M$ defines a diffeomorphism of a neighbourhood of the point $\beta(0)$ in the corresponding leaf onto a neighbourhood of the point $\beta(b)$ in the corresponding leaf. Moreover, it is not difficult to verify that such a curve defines a global diffeomorphism of the universal covering of the leaf passing through the point $\beta(0)$ onto the universal covering of the leaf passing through the point $\beta(b)$. The results of Hermann [13] ensure that these mappings send geodesics into geodesics, and therefore they are isometries for the induced Riemannian metrics on leaves. The set of orthogonal curves starting and ending in a given leaf L define a pseudogroup of local isometries of the leaf L which we denote by $H(L)$ and which we call the tangential holonomy pseudogroup of the leaf L. It is not difficult to prove that the tangential holonomy pseudogroup of different leaves are equivalent [12,1,2].

Let $K(L,x)$ be the subgroup of $\pi_1(L,x)$ of the homotopy classes of loops τ in L for which $\tau_\sharp\gamma = \gamma$ for any curve γ tangent to Q and starting at the point x. In [2], the authors notice that $K(L,x)$ is a normal subgroup of $\pi_1(L,x)$.

Let $\hat{L}_x$ be the covering of L corresponding to the subgroup $K(L,x)$, i.e. $\hat{L}_x$ is the quotient of the space $C(L,x)$ of paths in L starting at x by the equivalence relation defined as follows.

Let $\tau_1, \tau_2 : [0,1] \to L$ be two curves in the leaf L starting at the same point. We say that they are equivalent if they have the same end and the loop $\tau_1\tau_2^{-1}$ defines the class which belongs to the subgroup $K(L,x)$.

The group $\pi_1(L,x)$ acts on $\hat{L}_x$; let $\alpha = [\tau_0] \in \pi_1(L,x)$, then

$$\alpha[\tau] = [\tau_0\tau] \in \hat{L}_x.$$

It is not difficult to check that the kernel of the homomorphism

$$\pi_1(L,x) \to \mathrm{Diff}(\hat{L}_x)$$

defined above is equal to $K(L,x)$. The quotient group, isomorphic to the image of this homomorphism, $H_g(L,x) = \pi_1(l,x)/K(l,x)$, consists of the so-called deck transformations of the covering. This covering space of the leaf does not depend on the point and is called the tangent holonomy covering of L. The group $H_g(L)$ (we drop x) is a subgroup of the group of isometries of $\hat{L}$.

Proposition 1 [2]. *There exists a natural surjective homomorphism from the group $H_g(L)$ onto the holonomy group of the leaf L at x.*

This homomorphism is injective if the foliation is without holonomy [2], or the foliation is Riemannian [6].

Let us denote by $C(\mathcal{F}^{\perp}, x)$ the set of curves $\alpha : [0,1] \to M$ orthogonal to $\mathcal{F}$ and starting at x. Let η be a leaf curve starting at x (i.e. $\eta : [0,1] \to L$) and $\eta(1) = \alpha(1)$. This pair of curves defines a mapping: $\Phi_{(\alpha,\eta)}$: {leaf curves starting at x} $\ni \tau \mapsto \eta\alpha_{\sharp}\tau \in$ {leaf curves starting at x}. Cairns [6] demonstrated that the mapping $\Phi_{(\alpha,\eta)}$ induces a smooth mapping of $\hat{L}$. He denotes by S_x the set of all couples (α, η) of curves in M such that:

1. α is orthogonal to $\mathcal{F}$ and η is a leaf curve;
2. $\alpha(0) = \eta(0) = x$ and $\alpha(1) = \eta(1) \in L$.

Elements of S_x are called *serviettes* of $\mathcal{F}$ at x. The set admits a natural product structure such that

$$\Phi_{(\alpha_1,\eta_1)(\alpha_2,\eta_2)} = \Phi_{(\alpha_1,\eta_1)}\Phi_{(\alpha_2,\eta_2)}.$$

To get a group different from the fundamental group of the leaf we have to pass to the quotient of S_x by an equivalence relation. We say that two *serviettes* s_1 and s_2 are equivalent if they define the same automorphisms of the set {leaf curves starting at x}, up to $K(L,x)$. The quotient is denoted by Σ_x and it is called the group of *serviettes* of $\mathcal{F}$ at the point x.

It is not difficult to prove that up to isomorphism the group Σ_x does not depend on the point x of M [6, B.7]. Directly from the definition and the construction of the group Σ_x one can deduce that the mapping Φ defines an injective homomorphism $\bar{\Phi} : \Sigma_x \to \mathrm{Diff}(\hat{L}_x)$. About the image of this homomorphism one can prove the following theorem [6, B.8].

Theorem 1. *With the above notation:*

(a) *the image of the mapping $\bar{\Phi}$ is a subgroup Γ_x of the group of isometries of $\hat{L}_x$;*

(b) *the pseudogroup of tangential holonomy $H(L)$ is equivalent to the pseudogroup of local isometries of $\hat{L}_x$ generated by Γ_x.*

At the end let us stress that it is not difficult to verify that if the orthogonal bundle $T\mathcal{F}^{\perp}$ is integrable, then the foliation defined by it is Riemannian. Thus we have a kind of duality: Riemannian – totally geodesic; if the *tangent* is Riemannian, then the *orthogonal* is totally geodesic and vice versa.

1.2. Ehresmann connections

The concept of an Ehresmann connection appears for the first time, under a different name, in [8]. However, recent studies of totally geodesic foliations have brought out its importance [1,5,19] in geometry.

The notion of a foliation admitting an Ehresmann connection is a natural generalization of the concept of a totally geodesic foliation, namely:

Let $(M, \mathcal{F})$ be a foliated manifold. A supplementary subbundle Q to $T\mathcal{F}$ is called an Ehresmann connection if for any point $x \in M$ and any pair of curves $\alpha : [0,a] \to M$ and

$\beta : [0, b] \to M$ with the same starting point, such that the curve α is a leaf curve, and β is tangent to Q, there exists a smooth mapping $\sigma : [0, a] \times [0, b] \to M$ such that:

(1) the curves $\sigma_s : [0, a] \to M, \sigma_s = \sigma \mid [0, a] \times \{s\}, s \in [0, b]$, are leaf curves;

(2) $\sigma_0 = \alpha$;

(3) the curves $\sigma^t : [0, b] \to M, \sigma^t = \sigma \mid \{t\} \times [0, b], t \in [0, a]$, are tangent to Q;

(4) $\sigma^0 = \beta$.

In Section 1.1 we have noticed that the orthogonal bundle to a totally geodesic foliation on a complete Riemannian manifold is an Ehresmann connection for this foliation. By the duality Riemannian – totally geodesic the orthogonal bundle to a Riemannian foliation is also an Ehresmann connection for this foliation.

Very many results concerning Riemannian foliations were proved using the properties of geodesics under the assumption that the Riemannian manifold is complete. However, in most cases it is sufficient to know that the orthogonal subbundle is an Ehresmann connection and to demonstrate that it is an Ehresmann connection it is not necessary to assume that the Riemannian metric is complete. Let us give an example.

Example. Let $M = \mathbb{R}^3 - \{(0,0)\} \times \mathbb{R}$ and $p : M \to \mathbb{R} \times \mathbb{R}$ be the projection $(x, y, z) \mapsto (x, y)$. Then $p(M) = \mathbb{R}^2 - \{(0,0)\}$. With the standard metric g_0 on M induced from the euclidean space $\mathbb{R}^3$, the Riemannian manifold (M, g_0) is not complete; however, the orthogonal bundle to the foliation defined by the connected components of the fibres of the submersion p is an Ehresmann connection for this foliation. To put it succinctly, we can define and work with an Ehresmann connection when certain geodesics are defined *uniformly* along the leaves.

It is not difficult to verify that for foliations with an Ehresmann connection we can define the same objects and prove the "same" results as those mentioned in Section 1.1 for totally geodesic foliations.

1.3. Symplectic manifolds

Let (M, ω) be a symplectic manifold of dimension $2n$. A foliation $\mathcal{F}$ of dimension n is lagrangian if for any vectors X, Y tangent to $\mathcal{F}$, $\omega(X, Y) = 0$. For more information on symplectic manifolds and lagrangian foliations, see [24]. The leaves of a lagrangian foliation are affine manifolds [24,16]. However, in general, we do not know whether they are complete or not as affine (flat) manifold.

A connection ∇ on M is called symplectic if $\nabla\omega = 0$. This condition is equivalent to the fact that the connection ∇ is the extension of a connection in the $Sp(n)$-reduction $B(M, Sp(n))$, defined by the symplectic form, of the bundle of linear frames $L(M)$.

In [15] the author demonstrates that, cf. Theorem 1, given a supplementary lagrangian subbundle Q there exists a unique symplectic connection ∇, called bilagrangian, satisfying the following conditions:

(1) $\nabla T\mathcal{F} \subset T\mathcal{F}$ and $\nabla Q \subset Q$,

(2) $T(X, Z) = 0$ if $X \in T\mathcal{F}$, $Z \in Q$, where T is the torsion tensor of ∇,

(3) $T(X,Y) = -\pi_Q([X,Y]) = 0$, $X, Y \in T\mathcal{F}$,

(4) $T(X,Y) = -\pi_{\mathcal{F}}([X,Y])$, $X, Y \in Q$,

where π_Q and $\pi_{\mathcal{F}}$ are the orthogonal projections on Q and $T\mathcal{F}$, respectively.

The first condition ensures that the connection is, in fact, a connection in the $GL(n) \times GL(n)$-reduction of the linear frame bundle corresponding to the decomposition $TM = T\mathcal{F} \oplus Q$. Such a bilagrangian connection ∇ is torsion free and flat along $\mathcal{F}$ and flat along Q, thus only mixed components of the curvature can be non-zero. In [15] one can find explicit formulas for this connection.

Now let us assume that the supplementary subbundle Q is involutive. In this case the associated bilagrangian connection is torsion free and leaves of both foliations are affine manifolds. We say that the foliations are Heisenberg related [18] iff their tangent bundles are locally spanned by locally Hamiltonian vector fields $\omega^* \mathrm{d}q_i$ and $\omega^* \mathrm{d}p_i$, respectively, with the functions $q_i, p_i, i = 1, \ldots, n$, satisfying the canonical Poisson bracket rules: $[q_i, q_j] = 0 = [p_i, p_j]$, $[q_i, p_j] = \delta^i_j$, $i, j = 1, \ldots, n$.

Hess proved the following proposition.

Proposition 2. *If $\mathcal{F}$ and Q are Heisenberg related foliations iff the associated bilagrangian connection is flat.*

For simplicity, we prefer to work with the bilagrangian connection and not with the induced one on $T\mathcal{F}$. Similar results can be formulated and proved using the induced connection.

We call the bilagrangian connection associated with the subbundle Q tangential if

$$\Omega(X,Y) = 0 \quad \text{for } X \in T\mathcal{F} \text{ and } Y \in Q.$$

Let us consider ∇ as a connection in the $GL(n) \times GL(n)$-reduction of the linear frame bundle. Its connection form ω^∇ takes value in the Lie algebra $gl(n) \times gl(n)$. Thus ω^∇ can be decomposed into two components $\omega^\nabla = (\omega^1, \omega^2)$. Let ∇_1 and ∇_2 be the induced connections in the vector bundles $T\mathcal{F}$ and Q, respectively. The projections $\pi_{\mathcal{F}}$ and π_Q define the mappings of the corresponding bundles of frames:

$$\begin{array}{ccccc}
L(\mathcal{F}) & \xleftarrow{\;p_{\mathcal{F}}\;} & B(M, GL(n) \times GL(n)) & \xrightarrow{\;p_Q\;} & L(Q) \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_2} \\
M & \xleftarrow{\;\mathrm{id}\;} & M & \xrightarrow{\;\mathrm{id}\;} & M
\end{array}$$

where $L(\mathcal{F})$ and $L(Q)$ denote the bundles of linear frames of $T\mathcal{F}$ and Q, respectively. If ω_1 and ω_2 are the connection forms of ∇_1 and ∇_2, respectively, then $\omega^1 = p^*_{\mathcal{F}}\omega_1$ and $\omega^2 = p^*_Q\omega_2$ [28]. Since the Lie algebra is of the semi-diagonal form the structure equations split into

$$\Omega^i = d\omega^i + \tfrac{1}{2}[\omega^i, \omega^i], \qquad \Theta^i = d\theta^i + \omega^i \wedge \theta^i,$$

where $i = 1, 2$, $\Theta = (\Theta^1, \Theta^2)$, $\theta = (\theta^1, \theta^2)$, $\Omega = (\Omega^1, \Omega^2)$ and the forms $\Theta^1, \Theta^2, \theta^1, \theta^2$ take value in $\mathbb{R}^n$ but the forms Ω^1, Ω^2 in $gl(n)$. It is not difficult to show that $\Omega^2 = p_Q^* \Omega_2$ and $\Theta^2 = p_Q^* \Theta_2$ where Ω_2 and Θ_2 are the curvature and the torsion forms of the connection ω_2, respectively [28]. From the very definition of the torsion form and the properties 2 and 4 of the connection ∇, it is easy to obtain that $i_X \Theta^2 = 0$ for $X \in dp^{-1}(Q)$. Thus $i_X \Theta_2 = 0$ for $X \in dp_2^{-1}(Q)$ and for that matter for $X \in T\mathcal{F}_Q$.

On $L(Q)$ the forms θ_2 and $d\theta_2$ define a natural foliation $\mathcal{F}_Q$ of the same dimension as $\mathcal{F}$ and projecting by p_2 onto $\mathcal{F}$. For $X \in T\mathcal{F}_Q$ $i_X \Theta_2 = 0$ iff $i_X \omega_2 = 0$. Since $T\mathcal{F}_Q \subset dp_2^{-1}(Q)$, the foliation $\mathcal{F}_Q$ is horizontal and the connection ∇_2 is basic (or adapted in Bott's sense [3]). On the other hand we know that $\Omega(X, Y) = 0$ for $X, Y \in dp^{-1}(T\mathcal{F})$, so $\Omega_2(X, Y) = 0$ for $X, Y \in dp_2^{-1}(T\mathcal{F})$, and if we assume that the connection ∇ is tangential, then $i_X \Omega_2 = 0$ for $X \in T\mathcal{F}_Q$. We have just proved that the connection ω_2 is transversely projectible (or foliated) [28,20]. Moreover, our assumptions ensure that this connection is flat. Therefore we have demonstrated the following fact.

Proposition 3. *Let $\mathcal{F}$ be a lagrangian foliation on a symplectic manifold (M, ω). If $\mathcal{F}$ admits a supplementary lagrangian subbundle for which the associated bilagrangian connection is tangential, then the foliation $\mathcal{F}$ is transversely affine, i.e. admits a transversely projectible, flat, torsion free connection.*

1.4. Physical applications

To justify our interest in Ehresmann connections we would like to recall just one but important application. Lagrangian subbundles appear naturally in the quantization procedure. A careful look at the Kostant–Souriau quantization of classical observables reveals that it also depends on another supplementary lagrangian subbundle. Hess [15] improved this classical method using the bilagrangian connection described in Section 1.3. He claims to unify in this way several approaches to the quantization. In this paper we show that under some very natural assumptions the lagrangian foliation and the supplementary lagrangian subbundle have certain, very precise properties, which also influence the topology and geometry of the ambient symplectic manifold. It should have some important consequences for the quantization [15,18,21–23].

2. Main results

Assuming the completeness of leaves as affine manifolds we obtain the following theorem.

Theorem 2. *Let $\mathcal{F}$ be a lagrangian foliation on a compact connected symplectic manifold. Let Q be a supplementary lagrangian subbundle to $T\mathcal{F}$ for which the bilagrangian connection is tangential. If the leaves of $\mathcal{F}$ are complete affine manifolds then this subbundle Q is an Ehresmann connection for the foliation $\mathcal{F}$.*

The existence of an Ehresmann connection has some important consequences for the foliation [1,2].

Corollary 1. *Under the assumptions of Theorem 1 universal coverings of leaves are affinely diffeomorphic.*

Corollary 2. *Let $\mathcal{F}$ be a lagrangian foliation on a compact connected symplectic manifold. Let Q be an integrable supplementary lagrangian subbundle to $T\mathcal{F}$ for which the bilagrangian connection is tangential. If the leaves of $\mathcal{F}$ are complete affine manifolds then the universal covering $\tilde{M}$ of M is an affine product $\tilde{L} \times \tilde{K}$ where $\tilde{L}$ is the universal covering of leaves of $\mathcal{F}$ and $\tilde{K}$ is the universal covering of leaves of Q. Moreover, leaves of $\mathcal{F}$ and Q intersect one another.*

As a corollary of Theorem 1 we get [2]:

Corollary 3. *If $\mathcal{F}$ has a compact leaf L_0 with finite $H_Q(L_0, x_0)$, then every leaf L of $\mathcal{F}$ is compact with finite $H_Q(L, x)$.*

In our case Theorem 1 can be reformulated as follows.

Theorem 3. *Let $\mathcal{F}$ be a lagrangian foliation on a compact connected symplectic manifold M. Let ∇ be a complete tangential bilagrangian connection for $\mathcal{F}$. Then for any leaf L the tangential holonomy pseudogroup $\mathcal{H}(L)$ is equivalent to the pseudogroup generated by a subgroup Γ of the affine transformations of the tangential holonomy covering of leaves of $\mathcal{F}$.*

As we have said the leaves of a lagrangian foliation are affine manifolds. Therefore the fundamental group $\pi_1(L, x_0)$ of a leaf L of $\mathcal{F}$ admits two natural representations into the linear group $GL(n)$. The first one $\varphi : \pi_1(L, x_0) \to GL(n)$ is the linear representation of the germ holonomy representation of the leaf, i.e.

$$\varphi([\gamma]) = d_{x_0} h_\gamma,$$

where h_γ is the holonomy of the leaf L corresponding to the loop γ. The second one $\psi : \pi_1(L, x_0) \to GL(n)$ is just the linear holonomy representation of the affine manifold L [9]. Inaba [16] has proved that

$$\psi^t \circ \varphi = \mathrm{id}. \qquad (*)$$

This relation provides us with some conditions assuring the completeness of the affine structure of compact leaves. Let us formulate two theorems. In the first one we provide some known conditions which ensure the completeness of the affine structure of leaves. In the second one these conditions are transposed by the relation $(*)$.

Theorem 4. *Let $\mathcal{F}$ be a lagrangian foliation of a compact, connected, symplectic manifold. Assume that all leaves of $\mathcal{F}$ are compact. Then if one of the following conditions is satisfied:*

(i) *the leaves are $SO(p-1,1)$-affine;*
(ii) *the fundamental groups of leaves are nilpotent and the holonomy representations are irreducible;*
(iii) *the fundamental groups of leaves are nilpotent and they have parallel volume forms;*
(iv) *the fundamental groups of leaves are nilpotent and the developing maps are surjective;*
then any lagrangian subbundle supplementary to $\mathcal{F}$, for which the bilagrangian connection is tangential, is an Ehresmann connection.

The theorem is a direct consequence of Theorem 2. The completeness of the affine structure on leaves is assured by the results of [9,11].

If we take into account the relation $(*)$, then our considerations yield the following theorem.

Theorem 5. *Let $\mathcal{F}$ be a lagrangian foliation of a compact, connected, symplectic manifold. Assume that all leaves of $\mathcal{F}$ are compact. Then any lagrangian supplementary subbundle Q to $\mathcal{F}$ for which the bilagrangian connection is tangential is an Ehresmann connection if one of the following conditions is satisfied:*
(i) *the foliation $\mathcal{F}$ is an $SO(p-1,1)$-transversely affine foliation;*
(ii) *the fundamental group of the manifold M is nilpotent and the holonomy representation is irreducible;*
(iii) *the fundamental group of the manifold M is nilpotent and the foliation $\mathcal{F}$ has a parallel basic volume form;*
(iv) *the fundamental group of the manifold M is nilpotent and the Zariski closure of the affine holonomy group acts transitively.*

Remarks.
(1) If Q is a foliation, then we can replace the assumption about the associated bilagrangian connection by *the foliations are Heisenberg related* [15].
(2) Under the assumptions of Theorems 5 or 6, the universal coverings of leaves are diffeomorphic, cf. Corollary 1.
(3) The affine group Aff(n) is an algebraic group, i.e. it is a Lie group and an algebraic subset of $\mathbb{R}^{n^2}$. A Lie subgroup N of Aff(n) need not be an algebraic set. The Zariski closure of a Lie group N is the smallest algebraic subset of $\mathbb{R}^{n^2}$ containing N; it is, of course, a group. It is the closure of the subset N of Aff(n) in the Zariski topology – the topology in which the algebraic sets are the closed subsets.

Proof. Let us recall that the linear holonomy of a leaf can be read as the isotropy group of the affine holonomy group at a point of the corresponding orbit [9,30]. The condition (i) via the relation $(*)$ ensures that the leaves are $SO(p-1,1)$-flat manifolds, thus, according to [7], they are complete affine manifolds. Theorem 2 concludes the proof of this case.

According to the results of [29] the conditions (ii)–(iv) are equivalent. The relation $(*)$ ensures that the affine holonomy representations of leaves have values in $SL(p)$, thus the leaves admit a parallel volume form and their affine holonomy groups are nilpotent. Therefore they are complete affine manifolds and we can apply Theorem 2. □

It is tempting to articulate some relation between the Maslow class of a lagrangian foliation and the existence of an Ehresmann connection. A good testing ground would be one-dimensional foliations on the 2-torus. The Maslov class of such a foliation is equal to the difference between the number of "positively" oriented and the number of "negatively" oriented Reeb components. Therefore there exist foliations with Reeb components whose Maslov class is zero [14].

Let us consider a foliation with Reeb components but whose Maslov class is zero. Any transverse bundle being one-dimensional is integrable. This foliation has also Reeb components and one can easily check that some leaves of these two foliations do not intersect one another. This means that the foliation does not admit an Ehresmann connection [26].

On the other hand, foliations admitting an Ehresmann connection have no vanishing cycles [10] thus in the case of codimension 1 foliations they have no Reeb components. Therefore their Maslov class must be zero.

3. The proof of Theorem 2

Let us assume that the manifold M is compact and the connection ∇ is leafwise complete, i.e. the leaves are complete affine manifolds. In this case there exists $\varepsilon > 0$ such that for any $x \in M$ the ball $B(x,\varepsilon)$ is convex. Therefore as leaves are totally geodesic the ε-balls $B_L(x,\varepsilon)$ in any leaf L are equal to the corresponding connected component of $B(x,\varepsilon) \cap L$. Thus there exists $\varepsilon > 0$ such that the balls $B_L(x,\varepsilon)$ are convex; among other things it means that the injectivity radius for leaves is greater than ε.

To prove that the subbundle Q is an Ehresmann connection we need to show that any pair of curves (α,σ), α – tangent to a leaf, σ – tangent to Q, can be extended to a full rectangle; to be precise:

Let $\alpha : [0,a] \to M$ be a leaf curve and let $\sigma : [0,b] \to M$ be a curve tangent to Q such that $\alpha(0) = \sigma(0) = x$. We have to show that there exists a mapping $\kappa : [0,a] \times [0,b] \to M$, sometimes denoted by $\kappa_{\alpha,\sigma}$, called the rectangle defined by σ and α such that for any $t \in [0,a]$ $\kappa \mid \{t\} \times [0,b]$ is tangent to Q and for any $s \in [0,b]$ $\kappa \mid [0,a] \times \{b\}$ is tangent to $\mathcal{F}$. Let $\tilde{Q} = (\mathrm{d}p^{-1}(Q)) \cap \ker \omega^\nabla$.

First we will show it in the case of a geodesic tangent to a leaf. Let $\bar{x}$ be a point of B over x and $\tilde{\sigma}$ be the $\tilde{Q}$-lift of σ and $\tilde{\alpha}$ be the horizontal lift of α; it is an integral curve of some vector field $B(\xi)$, $\xi \in \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$. Using the flow of $B(\xi)$ we can fill the rectangle, namely $\tilde{\kappa} : [0,a] \times [0,b] \to B$ and $\tilde{\kappa} : [0,a] \times \{t\} = \varphi(-,\tilde{\sigma}(t))|[0,a]$ where φ is the flow of $B(\xi)$. We would like to prove that for any $X \in \tilde{Q}$ $[X, B(\xi)] \in \tilde{Q}$, which will ensure that the curves $\tilde{\kappa}^s$, $s \in [0,a]$, are tangent to $\tilde{Q}$. As our connection is bilagrangian, $\Theta(X, B(\xi)) = 0$. Let $X \in \tilde{Q}$ and take $B(\xi)$, $\xi \in \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$, then $0 = \Theta(X, B(\xi)) = (\Theta^1(X, B(\xi)), \Theta^2(X, B(\xi)))$. Thus $0 = \Theta^1(X, B(\xi)) = \mathrm{d}\theta^1(X, B(\xi)) + \omega^1 \wedge \theta^1(X, B(\xi)) = \mathrm{d}\theta^1(X, B(\xi)) = -\frac{1}{2}\theta^1([X, B(\xi)])$, as $\omega^1(B(\xi)) = 0$, $\omega^1(X) = 0$ and $\theta^1(X) = 0$ since $\tilde{Q} \subset \ker \theta^1$. Therefore indeed $[X, B(\xi)] \in \tilde{Q}$, for any $X \in \tilde{Q}$ and

$B(\xi), \xi \in \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$. Now, since $\Omega(X, B(\xi)) = 0$ the structure equations yield

$$\{\mathrm{d}\omega^\nabla + [\omega^\nabla, \omega^\nabla]\}(X, B(\xi)) = \mathrm{d}\omega^\nabla(X, B(\xi)) = -\tfrac{1}{2}\omega^\nabla[X, B(\xi)] = 0.$$

Thus $[X, B(\xi)] \in \ker \omega^\nabla$. Hence $[X, B(\xi)] \in \ker \theta^1 \cap \ker \omega^\nabla = \tilde{Q}$ for any $X \in \tilde{Q}$ and $B(\xi), \xi \in \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$.

Thus the vector fields $B(\xi)$ preserve the subbundle $\tilde{Q}$. Therefore for any $s \in [0, a]$ the curve $\tilde{\kappa} \,|\, \{s\} \times [0, b]$ is a curve tangent to $\tilde{Q}$. The mapping $\kappa = \pi\tilde{\kappa}$ is a rectangle we have been looking for. Let us remark that for any $t \in [0, b]$ the curve $\kappa \,|\, [0, a] \times \{t\}$ is a geodesic tangent to $\mathcal{F}$.

Let $\sigma : [0, b] \to M$ be any curve in M tangent to Q and let $\alpha' : [0, a] \to M$ be any leaf curve contained in $B_L(x, \varepsilon)$. As $\exp_{\sigma(t)} | B(0_{\sigma(t)}, \varepsilon)$ is a diffeomorphism on the image for any $t \in [0, b]$ we define our rectangle as follows.

We put $\kappa(s, t) = \kappa_{\alpha_s, \sigma|[0,t]}(s, t)$ where α_s is the geodesic in L linking x with $\alpha(s)$. Our previous considerations concerning the rectangle defined by a geodesic ensure that the mapping κ is, in fact, a rectangle we are looking for. Let us stress that the choice of ε was independent of the choice of σ.

Now let $\alpha : [0, a] \to M$ be any leaf curve in M. Then there exist points $0 = s_0 < s_1 < \cdots < s_m = a$ such that for any i $\alpha(s_{i_1}), \alpha(s_{i+1}) \in B(\alpha(s_{i-1}), \varepsilon)$. Let $\alpha_i = \alpha | [s_i, s_{i+1}]$. Let κ_1 be the rectangle corresponding to σ and α_1. The curve $\sigma_1 = \kappa | \{s_1\} \times [0, b]$ is tangent to Q and $\sigma_1(0) = \alpha_2(s_1)$. For σ_1 and α_2 we have a rectangle κ_2. After m steps we have our rectangle, namely $\kappa = \kappa_1 \cup \cdots \cup \kappa_m$.

Once again let $\alpha : [0, a] \to M$ be a leaf curve and $\sigma : [0, b] \to M$ be the Q-horizontal curve with $\alpha(0) = \sigma(0)$. For the rectangle κ defined by the pair α, σ, the curve $\alpha_\sharp\sigma$ is Q-horizontal; the curve $\sigma^\sharp\alpha$ is a leaf curve passing through $\sigma(b)$. From the construction of the rectangle one can easily notice that if α is a geodesic of ∇, so is the curve $\sigma_\sharp\alpha$.

We have seen that the curve σ defines a diffeomorphism from a neighbourhood of $\sigma(0)$ in the leaf L_0 onto a neighbourhood of $\sigma(b)$ in the leaf L_1 passing through these points. This mapping is called the holonomy along σ. It maps geodesics into geodesics, so it is a local affine transformation. Therefore we have proved the following lemma (cf. Section 5 of [2]):

Lemma 1. *Any lagrangian subbundle Q supplementary to the foliation $\mathcal{F}$ preserves the connection ∇.*

This lemma ensures that Corollary 1 is true. Therefore we have completed the proof of Theorem 1.

Final remarks.

(1) In Theorem 5 we have assumed that the leaves of the foliation are compact in order to prove that the affine structures of leaves are complete. If we drop this assumption using our method we cannot prove that transverse lagrangian subbundle is an Ehresmann connection. However, our foliation is a transversely complete transversely affine foliation [29]. Such foliations have some very interesting properties [27,29,30].

(2) Combining the methods developed by Cairns in [5] with ours it is not difficult to prove the existence of Ehresmann connections for foliations admitting adapted connections satisfying similar conditions.

References

[1] R.A. Blumenthal and J.J. Hebda, Complementary distributions which preserve the leaf geometry and applications to totally geodesic foliations, Quart. J. Math. Oxford 35 (1984) 383–392.
[2] R.A. Blumenthal and J.J. Hebda, Ehresmann connections for foliations, Indiana Univ. Math. J. 33(4) (1984) 597–611.
[3] R. Bott, Lectures on characteristic classes and foliations, SLN in Math. 279 (1972) 1–94.
[4] G. Cairns, Feuilletages totalement géodésiques, Śeminaire de Géométrie Differentielle, USTL Montpellier 1983–84, 12–36.
[5] G. Cairns, A general description of totally geodesic foliations, Tohoku Math. J. 38 (1986) 37–55.
[6] G. Cairns, Feuilletages géodésibles, Thèse, USTL Montpellier (1987).
[7] Y. Carrière, Autour de la conjecture de L. Markus sur les variétés affines, Invent. Math. 95 (1989) 615–628.
[8] Ch. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, Colloq. Topologie Bruxelles (1950) 29–55.
[9] D. Fried, W. Goldman and M.W. Hirsch, Affine manifolds with nilpotent holonomy, Comm. Math. Helv. 56 (1981) 487–523.
[10] C. Godbillon, *Feuilletages*, Progress in Mathematics (Birkhäuser).
[11] W. Goldman and M.W. Hirsch, Affine manifolds and orbits of algebraic groups, Trans. A.M.S. 295(1) (1986) 175–198.
[12] A. Haefliger, Leaves closures in Riemannian foliations, *A fête of topology* (Academic Press, Boston, MA, 1988) pp. 3–32.
[13] R. Hermann, A sufficient condition that a map of Riemannian manifolds be a fibre bundle, Proc. A.M.S. 11 (1960) 236–242.
[14] B. Herrera and G. Hector, preprint.
[15] H. Hess, Connections on symplectic manifolds and geometric quantization, Lecture Notes in Mathematics, Vol. 836 (Springer, Berlin) pp. 153–165.
[16] T. Inaba, The tangentially affine structure of lagrangian foliations and the tangentially projective structure of legendrian foliations, preprint.
[17] D.L. Johnson and A.M. Naveira, A topological obstruction to the geodesibility of a foliation of odd dimension, Geom. Dedicata 11 (1981) 347–352.
[18] B. Kostant, On the definition of quantization, in: *Coll. Int. du CNRS. Géométrie symplectique et physique mathématique*, Aix-en-Provence 1974, CRNS 1976.
[19] P. Molino, *Polarization*, Travaux en Cours (Herman, Paris).
[20] P. Molino, *Riemannian Foliations*, Progress in Math. Vol. 73 (Birkhäuser, 1988).
[21] D.J. Simms and N.M.J. Woodhouse, *Lectures on Geometric Quantization*, SLN in Phys., Vol. 53 (1976).
[22] J.-M. Souriau, *Structure des systèmes dynamiques* (Dunod, Paris, 1969).
[23] A. Weinstein, Symplectic manifolds and their lagrangian submanifolds, Adv. Math. 6 (1971) 329–346.
[24] A. Weinstein, *Lectures on Symplectic Manifolds*, CBMS Regional Conference Series, Vol. 29 (AMS, 1977).
[25] R. Wolak, Foliations admitting transverse systems of differential equations, Comp. Math. 67 (1988) 89–101.
[26] R. Wolak, Le graphe d'un feuilletage admettant un système d'équations différentielles, Math. Z. 201 (1989) 177–182.
[27] R. Wolak, The structure tensor of a transverse G-structure on a foliated manifold, Boll. U. M. I. 4 (1990) 1–15.
[28] R. Wolak, Transversely affine foliations compared with affine manifolds, Quart. J. Math. Oxford 41 (1990) 369–384.
[29] R. Wolak, Closures of leaves in transversely affine foliations, Canad. Math. Bull. 34(4) (1991) 553–558.
[30] N.M.J. Woodhouse, *Geometric Quantization*, Oxford Math. Monograph (Clarendon Press, Oxford, 1980).